The Josephus Problem

Introduction

The Josephus problem is based around Josephus Flavius; a Jewish soldier and historian who inspired an interesting set of mathematical problems. In 67 C.E., Josephus and 40 fellow soldiers were surrounded by a group of Roman soldiers who were intent on capturing them. Fearing capture, they decided that they would kill themselves instead and whilst Josephus did not agree with this proposal, he was afraid to disagree. Instead, he suggested that they arrange themselves in a circle and, counting around the circle in a clockwise direction, every second man should be killed until there was only one survivor, who would then kill himself. Josephus wanted to be that man (so he could survive!) and he, therefore, had to figure out which position to stand in. Identifying this position is the problem we consider today.

Aim of Workshop

The aim of this workshop is to introduce the students to combinatorics and pattern recognition via the Josephus problem. Students will also be provided with the opportunity to simulate the Josephus problem in the hope of discovering the general formula themselves.

Learning Outcomes

By the end of this workshop, students will be able to:

- Recognise numerical patterns
- Solve problems using powers of two
- Derive the general formula for the Josephus problem
- Calculate the winning position for the Josephus problem

Materials and Resources

One activity sheet per student (with one third of the class on Activity 1A, one third on Activity 1B and one third on Activity 1C).

KEY WORDS

Combinatorics
A branch of mathematics that studies combinations of objects taken from a finite set

The Josephus Problem: Workshop Outline

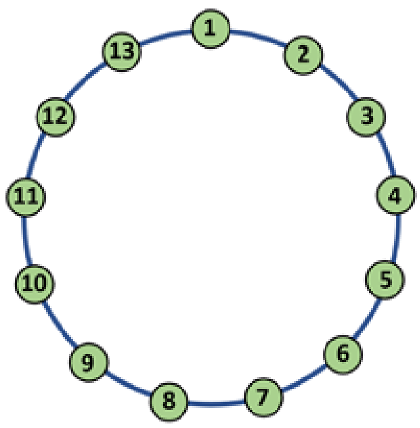
Suggested Time (Total mins)	Activity	Description
5 mins (00:05)	Introduction to the Josephus Problem	· Introduce the Josephus problem (see **Workshop Introduction**) · Demonstrate the idea with 3 volunteers
5 mins (00:10)	Class Activity The Josephus Problem	· Divide students into groups and ask them to apply the Josephus problem with their group - don't try to predict the winning position just yet (see **Appendix – Note 1** for example) · You may wish to go through a full example with the students
15 mins (00:25)	Activity 1 The Josephus Problem	· Hand out one activity sheet per student with one third of the class on Activity 1A, one third on Activity 1B and one third on Activity 1C · **Activity Sheet 1**: Students are asked to find out what position they should be in to survive in each of the given circles (see **Appendix - Note 2**)
15 mins (00:40)	Deduce the formula	· Collate the information into a table on the whiteboard. Have students spotted a pattern? · Discuss the ideas with the class · Deduce the formula using student input (see **Appendix – Note 3**)
10 mins (00:50)	Proof by induction (Optional)	· Explain how the formula could be proved by induction (see **Appendix – Note 4**)
5 mins (00:55)	Conclusion	· Briefly recap the problem, showing how the formula can be used to speed up the process · Encourage students to now find the solution to the original Josephus problem with 41 soldiers
5 mins (01:00)	Class Activity (Optional)	· Everybody in the room stands in a circle. Who survives now?

The Josephus Problem: Workshop Appendix

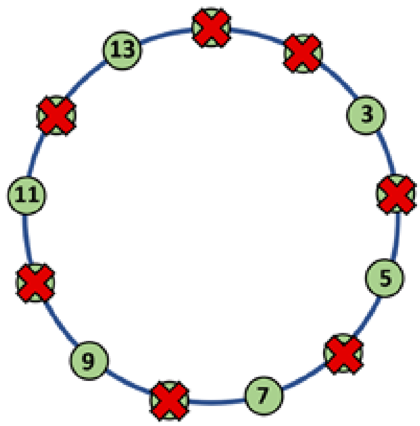
Note 1: Example of the Josephus Problem for 13 soldiers

(A round ends after the person in position n is killed or kills the person beside them)

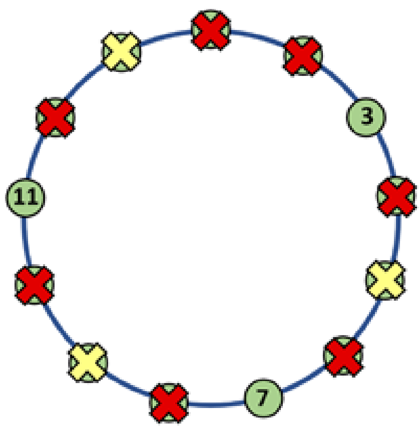
Josephus Problem Example (n=13)



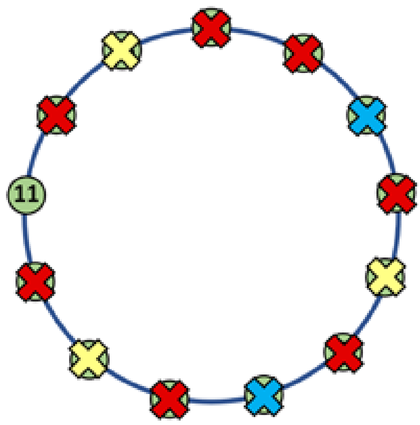
Since every second person is killed in this version of the Josephus problem, we want to find the position of survival. In this example, there are 13 people in the circle.



First round: Starting at 1, moving in a clockwise direction, we kill every second person by crossing them out as shown in red. 1 kills 2, 3 kills 4, 5 kills 6,…,13 kills 1.



Second round: The second round of killings is shown in yellow. This time 3 kills 5, 7 kills 9, and 11 kills 13.



Third round: The final round of killings is shown in blue. 3 kills 7, and 11 kills 3. 11 is, therefore, the position of survival (i.e. the last green circle).

Note 2: Solutions for Activity 1

There are 3 different versions of Activity Sheet 1, each containing circles of varying numbers of soldiers. Each student should work through one of these activity sheets, with the class divided into three groups so that all worksheets are completed.

Each activity sheet has circles of 4, 8, 16 and 32 soldiers to help students identify that a circle with a power of 2 has a '*winning (or survivor) position*' of 1. The rest of the activity sheets are made up of circles containing between 5 and 25 soldiers. When the answers from the three worksheets are collated in a table, as shown below, a pattern begins to emerge (see **Note 3** for formula).

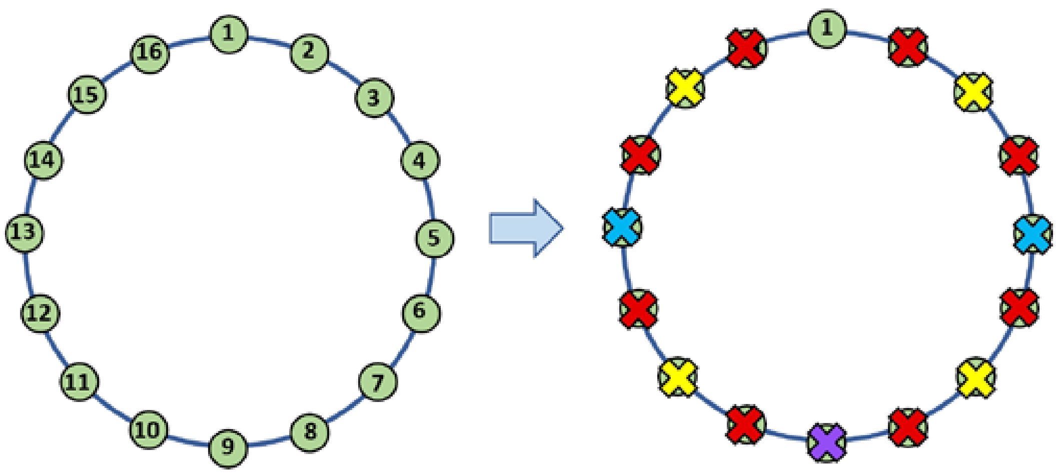
Number of Soldiers	Who Stays Alive?	Number of Soldiers	Who Stays Alive?
4	**1**	**16**	**1**
5	3	17	3
6	5	18	5
7	7	19	7
8	**1**	20	9
9	3	21	11
10	5	22	13
11	7	23	15
12	9	24	17
13	11	25	19
14	13	...	...
15	15	**32**	**1**

Note 3: Deducing the Formula

The best way to find the general solution to the Josephus problem is to map out some scenarios. In this workshop, we have started with 1, 2, 3, 4, 5, …, 10 and once we map the winning position in each case, a pattern begins to emerge.

Number of Soldiers	Winning Position
1	**1**
2	**1**
3	3
4	**1**
5	3
6	5
7	7
8	**1**
9	3
10	5

As indicated in the table above, none of the *winning positions* are even numbers. This is due to the fact that all the people in even-numbered positions are killed in the first round. We also notice that the winning position resets (that is, goes back to 1) for 1, 2, 4, and 8 soldiers. Whilst it intuitively makes sense for 1 to be the *winning position* when there are only 1 or 2 people in the circle, it may not be immediately clear for 4 or 8. However, 4 and 8 are both powers of 2 so we can check whether it resets at each *power* of 2. When we perform the Josephus elimination for 16 (2^4) soldiers, for example, 1 is again the *winning position*. The same is true for 32 (2^5) soldiers and this holds for 2^n soldiers.



To find the general formula, we can use the fact that any number can be written as a power of 2 plus a remainder (r).

Note: after r steps, whoever's turn it is will be the winner as we will be left with a power of 2.

When there are 5 soldiers, the winning position is 3.

If we subtract the next largest $2^n \leq 5$ from the number, we get:

$5 - 2^2 = 1$ remainder (r)

$2(1) = 2$ (Since we kill every second person, we multiply r by 2 to find the position in the circle after r killings)

$2 + 1 = 3$ (we add 1 as we are beginning from the 1^{st} position rather than the 0^{th} position)

3 is the winning position

When there are 20 soldiers, the winning position is 9. $20 - 2^4 = 4$ remainder $2(4) = 8$ $8 + 1 = 9$ 9 is the winning position	**The general formula is thus:** $N - 2^n = r$ (remainder) $2(r) = 2r$ $2r + 1$ is the winning position

Note 4: Proof by Complete Induction

The formula for the Josephus problem can be proved using proof by complete induction. Whilst this is outside the bounds of this workshop, you may wish to discuss it with your students.

We want to prove that if you have N people sitting in a circle and they take turns eliminating the next person in the circle until only one person remains, the person that remains will be in position 2r + 1, where r is the remainder upon division of N by the largest power of 2 less than or equal to N. We proceed by complete induction on N.

1. First of all, we must prove it for $N = 1$. Clearly, if there is only one person, they will be the only survivor. Here, $2^0=1$ is the largest power of 2 smaller than or equal to N and thus $r = 0$ and so $2(0) + 1 = 1$

2. We assume that this result is true for $N = 1, 2, 3,..., k - 1$. This is our inductive hypothesis (this is what we mean by Complete Induction).

3. We now need to show that this statement is true for k. To do this, we must consider two possibilities: first of all, when k is even, and secondly when k is odd.

Case 1:

If k is even, it can be expressed as $2^m + r$ where r is even and $r < 2^m$ (here we split k into the largest power of 2 less than k and the remainder part which must be even since k and 2^m are even).

After the first round of elimination, we will have that all even numbered people are eliminated, and we will be back at person number 1. We can consider this to be a new circle with $2^{m-1} + r/2$ people, still starting at 1 and having the same winning person as a circle that size, but with only odd numbered labels for the people in the circle.

By our inductive hypothesis, the survivor is in position $2\,(r/2) + 1 = r + 1$ in this new circle. This corresponds to position $r + (r + 1) = 2r + 1$ in the original circle as we had eliminated r people in positions before the $(r + 1)$ person in the new circle in the first round.

This proves the result for even k.

Case 2:
If k is odd, it can be expressed as $2^m + r$ where r is odd and $r < 2^m$ (again, splitting k into the largest power of 2 less than k plus the remainder part which must be odd since k is odd and 2^m is even).

This time, after the first round of elimination we will have that all even numbered people are eliminated, and the final, odd-numbered person has eliminated number 1. We can consider this to be a new circle only with having $2^{m-1} + (r - 1)/2$ people, starting at 3 (since 1 and 2 are eliminated) and only having odd numbered labels for people in the circle.

Again, by our inductive hypothesis, the survivor is in position $2((r - 1) / 2)$ in this new circle. This corresponds to the position $(r + 1) + r = 2r + 1$ in the original circle as we have eliminated r people in even positions and 1 person in an odd position (namely person number 1), in the positions before the r^{th} person in the new circle in the first round. This proves the result for odd k.

We can now apply complete induction to conclude that the winning position for any $N = 2^m + r$, where $r < 2^m$, is position $2r + 1$.

Sources and Additional Resources

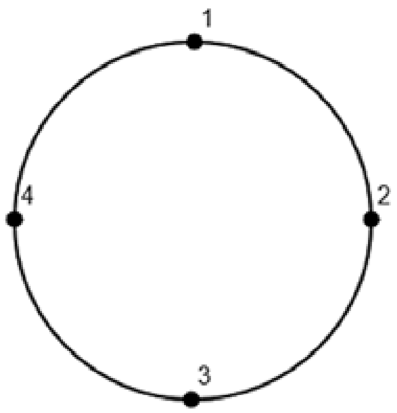
https://www.exploringbinary.com/powers-of-two-in-the-josephus-problem/ (Josephus problem)

https://www.youtube.com/watch?v=uCsD3ZGzMgE (Josephus problem general formula)

The Josephus Problem: Activity 1A

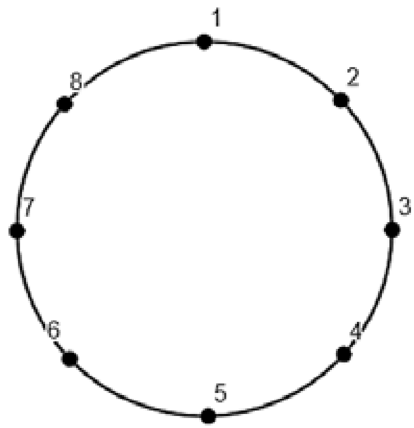
Q1. Find out who stays alive in each of the following scenarios by crossing off every second person in the circle. You may wish to use a different colour for each round (Note: A round ends after the person in position n is killed or kills the person beside them).

A



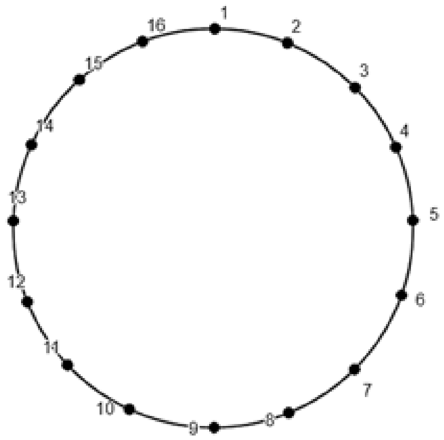
Who stays alive?

B



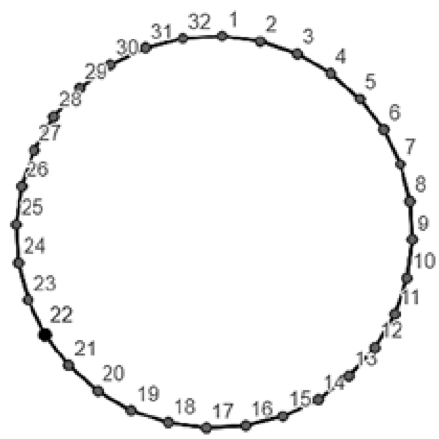
Who stays alive?

C



Who stays alive?

D

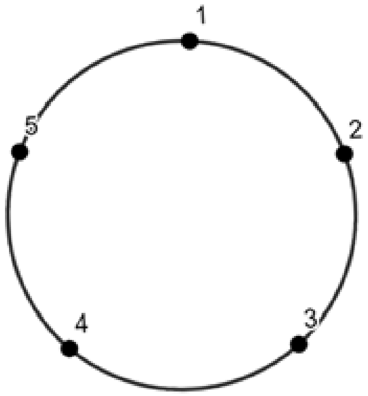


Who stays alive?

Q2. On each diagram A to D rewrite the number of people in the circle in the form 2^n. (Hint: $4 = 2^2$). Can you see a pattern?

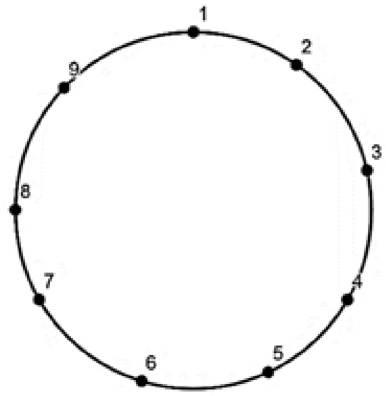
Q3. Please repeat Q1 using the following circles.

E



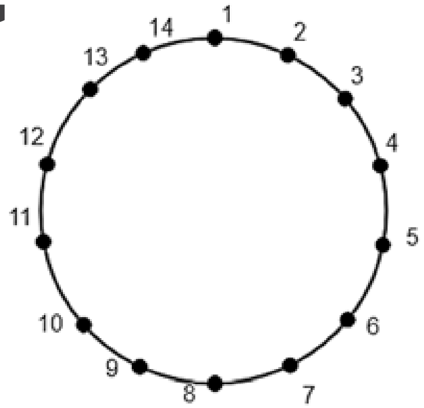
Who stays alive?

F



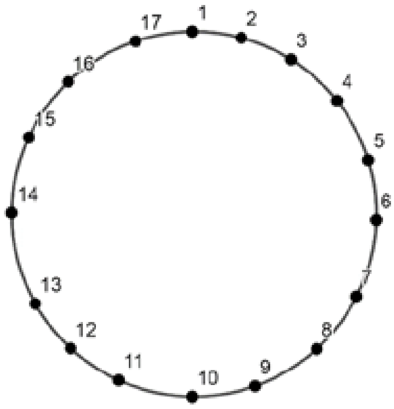
Who stays alive?

G



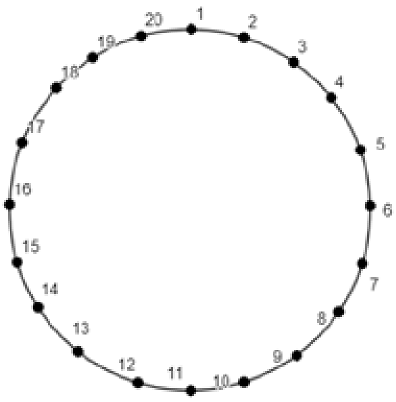
Who stays alive?

H



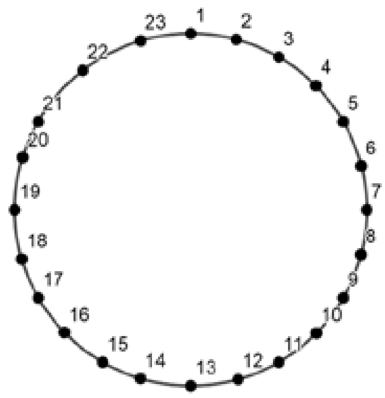
Who stays alive?

I



Who stays alive?

J

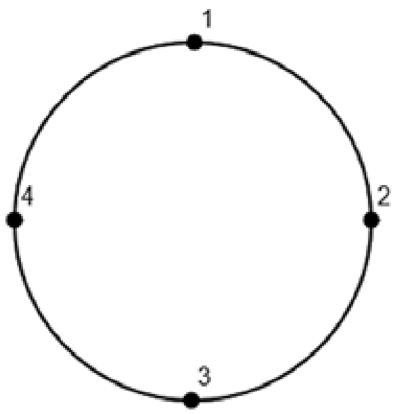


Who stays alive?

The Josephus Problem: Activity 1B

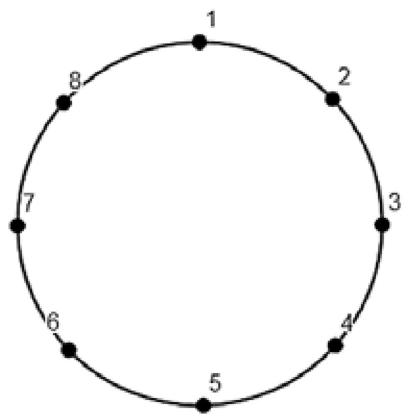
Q1. Find out who stays alive in each of the following scenarios by crossing off every second person in the circle. You may wish to use a different colour for each round (Note: A round ends after the person in position n is killed or kills the person beside them).

A



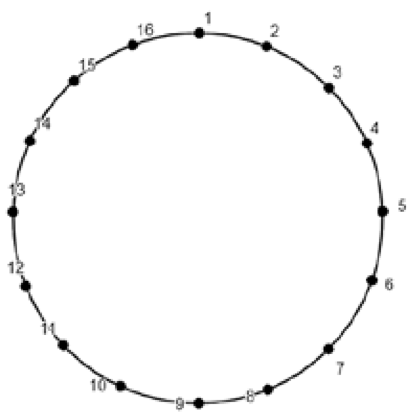
Who stays alive?

B



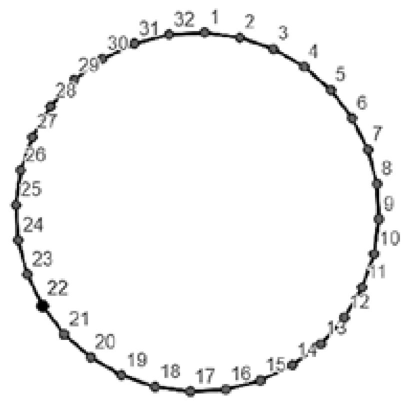
Who stays alive?

C



Who stays alive?

D

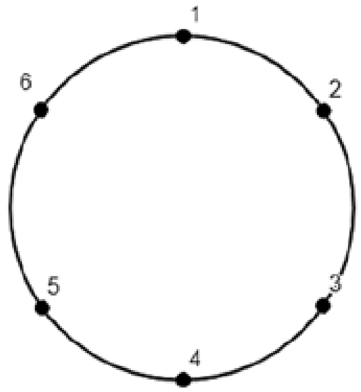


Who stays alive?

Q2. On each diagram A to D rewrite the number of people in the circle in the form 2^n. (Hint: $4 = 2^2$). Can you see a pattern?

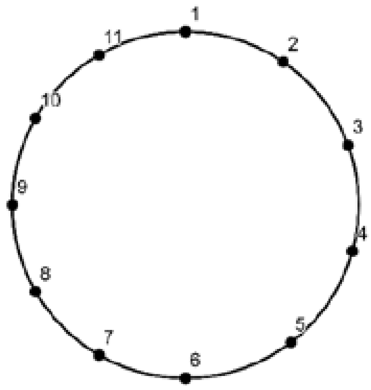
Q3. Please repeat Q1 using the following circles.

E



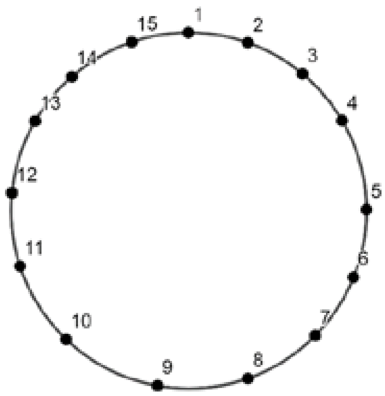
Who stays alive?

F



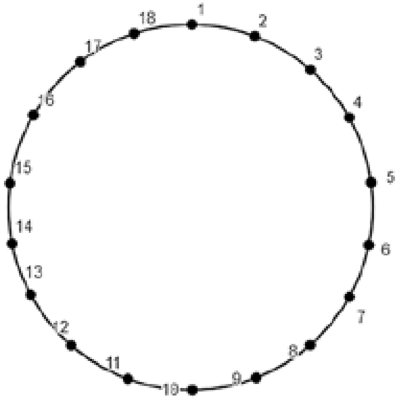
Who stays alive?

G



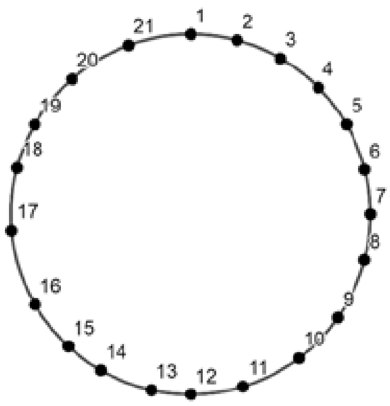
Who stays alive?

H



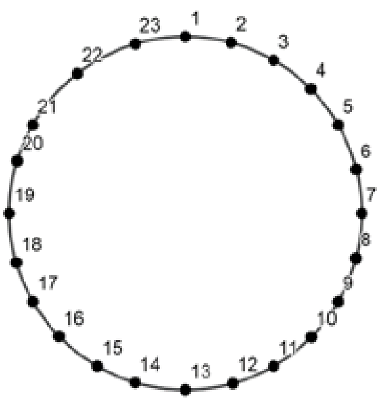
Who stays alive?

I



Who stays alive?

J

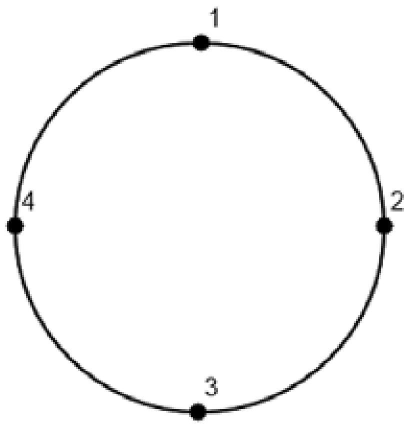


Who stays alive?

The Josephus Problem: Activity 1C

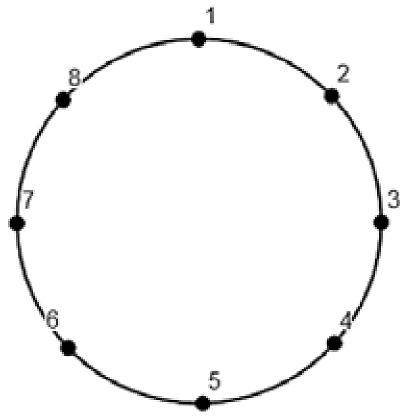
Q1. Find out who stays alive in each of the following scenarios by crossing off every second person in the circle. You may wish to use a different colour for each round (Note: A round ends after the person in position n is killed or kills the person beside them).

A



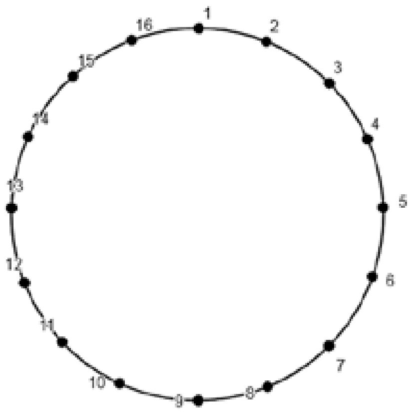
Who stays alive?

B



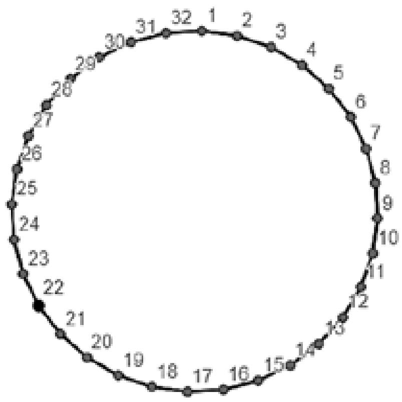
Who stays alive?

C



Who stays alive?

D

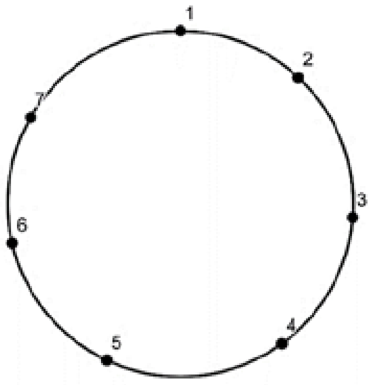


Who stays alive?

Q2. On each diagram A to D rewrite the number of people in the circle in the form 2^n. (Hint: $4 = 2^2$). Can you see a pattern?

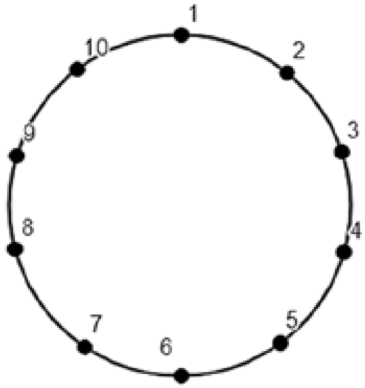
Q3. Please repeat Q1 using the following circles.

E



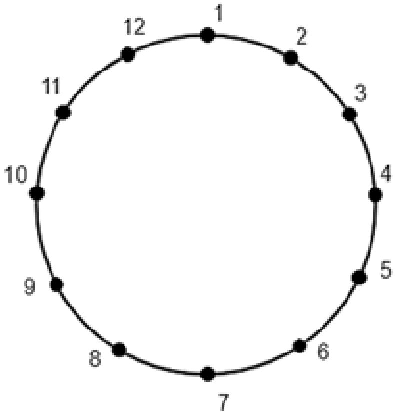
Who stays alive?

F



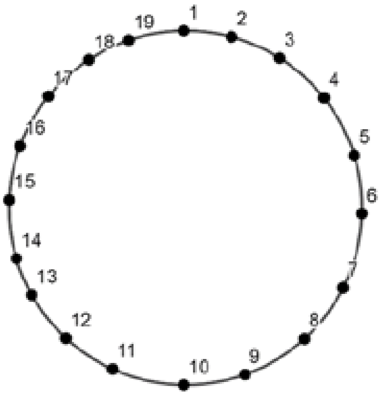
Who stays alive?

G



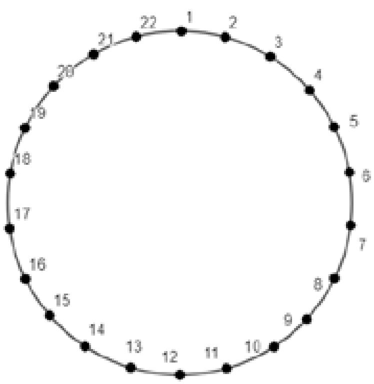
Who stays alive?

H



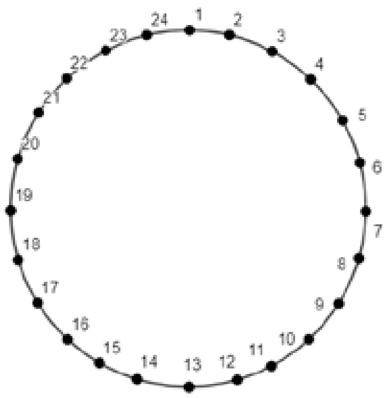
Who stays alive?

I



Who stays alive?

J



Who stays alive?